SECONDARY TEACHERS' CONCEPTIONALIZATIONS OF THE RELATIONSHIPS BEWEEN MATHEMATICAL MODELING AND PROBLEM SOLVING

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The purpose of this study is to gain understanding of how secondary teachers conceptualize the relationships between mathematical modeling and problem solving. Eight secondary teachers participated in semi-structured, think-aloud individual interviews. Some conceptualizations include (a) modeling and problem solving are two distinct processes; (b) modeling is a subset of, or a tool for, problem solving; (c) the two processes share similar characteristics such as a real-life scenario but are different in terms of whether a single correct answer exists; (d) the two processes are inseparable, completely enmeshed in each other, and co-dependent; and (e) problem solving is a reduced process of modeling. Teachers' conceptualizations are related to their preferred instructional sequence and the types of problems or activities they'd rather use.

Keywords: Modeling, Problem Solving, Teacher Knowledge, Teacher Educators

Mathematical modeling, or responding to real-world problems mathematically, meets the needs and interests of 21st century learners by offering abundant opportunities for developing critical skills such as adaptability, systems thinking, nonroutine problem solving, and complex communication skills (Bybee, 2010). As an example, Model Eliciting Activities (MEAs) are open-ended and context-rich activities that challenge learners to generate models or systems as useful solutions for complex real-world situations (Aguilar, 2021; Lesh & Doerr, 2003). Lesh and colleagues (e.g., Lesh & Harel, 2003; Lesh & Lehrer, 2003) implemented MEAs with students who were enrolled in remedial mathematics and were from poor, large urban school districts. Learners in these studies were competent in developing conceptual tools as they simultaneously built "communities of mind" and "the thinking of teams" (p.187). Despite the potential of mathematical modeling for actively engaging and motivating diverse learners, there is evidence that teaching of mathematical modeling is limited in P-12 classrooms (Doerr, 2007; Zbiek, 2016; Zbiek & Conner, 2006).

Purposes of Study

It has been difficult for the mathematical modeling community to come to an agreement on (a) a unified definition of mathematical modeling, (b) the characterization of the modeling process, or (c) how modeling is differentiated from traditional application problems. This lack of agreement has become one of the major challenges for teaching, learning, and research of modeling in the context of P-12 mathematics education (Cai et al., 2014).

The boundary between problem solving and modeling has never been clear. Some researchers (e.g., Blum & Niss, 1991; Lesh & Doerr, 2003) emphasize the differences between mathematical modeling and *traditional* problem solving and avoid using the term "problem solving" without qualification. These researchers often point out that the facts and rules in traditional application problems are restricted artificially so that the problems can be solved with readily available algorithms or basic operations. Therefore, traditional application problems are

also called “dressed up” word problems (Blum & Niss, 1991) or “pre-modeled problems” (Burkhardt & Pollak, 2006). While traditional problem solving often requires one cycle of straightforward interpretation of a real-world scenario, mathematical modeling requires multiple cycles of adapting, modifying, and refining ideas (Lesh & Harel, 2003).

Other researchers such as Selden et al. (1999) call all non-routine or novel problems that require higher-order reasoning simply “problems,” and routine problems that can be solved using clear procedures “exercises”. In their conception of problems, mathematical modeling *is* problem solving, whereas traditional problem solving is just an exercise.

Although we have some knowledge of how researchers perceive the relationship between mathematical modeling and problem solving, limited studies focused on teachers’ perspectives and how their perspectives might help explain their instructional practices and decisions. The purpose of this study is to gain understanding of teachers’ perspectives of the relationship between mathematical modeling and problem solving. Two research questions guide this study:

(1) How do secondary teachers conceptualize the relationships between mathematical modeling and problem solving? (2) How are teachers’ conceptualizations related to their perceptions of teaching mathematical modeling in mathematics classrooms?

Perspectives

Doerr and English (2003) define modeling as a process to develop “systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behaviour of some other familiar system” (p. 112). Several common and essential features of the modeling process include (a) modeling is a cyclic process that usually requires multiple iterations, (b) the process typically begins with a real situation, (c) modeling typically ends with the report of a successful result or the decision to revise the initial model to achieve a better result, and (d) the entire modeling process contains several common steps such as formulation, computation, interpretation, and validation (Blum & Leiß, 2007; Blum & Niss, 1991; Galbraith & Stillman, 2006; Pollak, 2003; Zbiek & Conner, 2006). Early depictions of modeling are often rooted in applied mathematics and do not differentiate between the modeling process and the problem-solving process (Burkhardt & Pollak, 2006).

However, more recent characterizations of the modeling process tend to point out the various distinctions between traditional application problems and modeling. Not all applied mathematical problem solving qualifies as modeling. This is because traditional problem solving usually takes a single cycle from givens to goals; whereas modeling tasks involve iterative cycles during which the emerging/initial model is subject to refinement, revision, adaptation, and modification (Dossey, 2010; Lesh & Doerr, 2003; Lesh & Harel, 2003; Lesh & Yoon, 2007; Lesh & Zawojewki, 2007). Solving traditional application problems may also be considered a reduced process of the modeling cycle (Blum & Niss, 1991; Lesh & Doerr, 2003; Lesh & Yoon, 2007). In MEAs, making symbolic descriptions of meaningful real problem situations, or *mathematization*, is at the heart of the tasks. However, in traditional application problems, students make meaning of symbolically described situations (Lesh & Doerr, 2003; Lesh & Harel, 2003). In other words, traditional application problems have already mathematized the situations for students.

In addition, traditional application problems and modeling often lead to very different products or outputs. Many traditional problems typically require a short answer in the form of a sentence or a number. On the other hand, for modeling tasks, students create conceptual tools and artifacts such as new techniques, examples, systems, approximations, and algorithms.

(Dossey, 2010; Lesh & Doerr, 2003; Lesh & Harel, 2003; Pollak, 2003)The criteria for judging the quality of modeling and traditional problem solving tend to be different as well. The solution to an application problem is typically judged based on correctness.

The criteria for judging a mathematical model may include reusability, modifiability, and shareability, as well as model's generalizability beyond the specific problem situation (Doerr, 2016; Lesh & Harel, 2003; Lesh & Yoon, 2007; Zawojewski, 2013). Lesh and Zawojewki (2007) have pointed out the "end in view" nature of MEAs, where models are judged based on the expressed needs of the client given at the beginning of the modeling task.

One subtle difference between mathematical modeling and solving traditional application problems is reflected in the relation between the modeling process and the world outside of mathematics. A modeling task arises from the real world. Modeling is a process to *mathematize the real world*, i.e., bringing the real world into contact with mathematics. On the other hand, a traditional problem is a process of *realizing mathematics*, i.e., given the mathematical knowledge and problem-solving strategies and heuristics, apply them to solve real world problems (Lesh & Doerr, 2003; Lesh & Lehrer, 2003). The best characterization of this difference is given by Dossey (2010), who states:

Modeling involves standing outside mathematics and looking into mathematics to find things that conceivably might help resolve the driving question. Applications, on the other hand, come from standing inside mathematics and noting that particular pieces can be used to better understand or highlight objects outside of mathematics. (p. 88)

Finally, Models and Modeling Perspectives (MMP) proposed by Lesh and colleagues (e.g., Lesh & Doerr, 2003; Lesh & Harel, 2003; Lesh & Zawojewki, 2007) is in direct contrast with the traditional view of the relationships between modeling, applied problem solving, and traditional problem solving. In the traditional view, applied problem solving is treated as a subset or special case of traditional problem solving. Within this view, modeling is a type of applied problem solving. In contrast, MMP treats traditional problem solving as a subset or a special case of "applied problem solving as modeling activities" (Lesh & Yoon, 2007, p.783).

Methods

Participants and Contexts

Participants included eight secondary (6-12) math teachers (Alexis, Alicia, Ann, Brian, Carrie, Eric, Katherine, Sarah, pseudonyms) who taught at eight different schools, and their ages ranged from mid-20s to late-40s. Prior to this study, the eight teachers were enrolled in a two-year secondary teacher preparation program (6-12) at a public university located in the Southeastern United States. The program was a non-traditional program designed for those who have a bachelor's degree in a content area. Half of the eight teachers held a bachelor's degree in an economics or business field: Katherine (Business Management), Ann (Accounting), Sarah (Finance), and Eric (Econometrics). Brian held a bachelor's degree in Aeronautics. The rest of the participants (Alicia, Alexis, and Carrie) majored in mathematics.

Prior to this study, the eight teachers participated in another study (about 1.5 years before this study) during which they had developed conceptual understanding of (a) the meanings of mathematical modeling, (b) the modeling cycle, and (c) criteria for judging the products of mathematical modeling. The eight teachers also solved one MEA and analyzed two sets of MEAs together during the earlier study. This study engaged the teachers in reasoning about the

relationships between problem solving and mathematical modeling, an aspect that was treated briefly and given only tentative interpretations in the earlier study.

Data Collection

Alexander and Dochy's (1995) graphic catalyst (displayed in Figure 1) was adapted to elicit the eight teachers' conceptualizations of the relationship between mathematical modeling and problem solving. Teachers were asked to indicate which option best represented their understanding of the relationship between modeling and problem solving. In addition, teachers were free to create their own model of this relationship as an alternative. Alexander and Dochy's categories: *separate*, *overlapping*, *inseparable*, *knowledge subsumption*, and *belief subsumption*, along with their meanings, were borrowed directly for this study except that the last two categories were changed to *modeling subsumption* and *problem-solving subsumption* according to the purpose of this study.

The meanings of these categories were presented to the teachers using the exact words from Alexander and Dochy (1995): (a) *Separat*e (Option 1) shows modeling and problem solving are "two distinct and unrelated entities;" (b) *Modeling subsumption* (Option 2) suggests that modeling is a component of problem solving; (c) *Problem-solving subsumption* (Option 3) shows modeling "as embedded within" problem solving; (d) *Inseparabl*e (Option 4) means modeling and problem solving are "completely overlapping and indistinguishable constructs;" and finally,

(e) *Overlapping* (Option 5) indicates "some dimensions of modeling are integrated with problem solving, still allowing for some aspects of each construct to remain separate and distinct" (p. 417, 419). "Problem solving" was intentionally left ambiguous or unqualified to elicit potentially diverse responses from the teachers. In addition, although each of the graphical representations in Figure 1 and their associated meanings were provided at the beginning of the study, teachers were also free to give their own interpretations.

Each of the eight teachers participated in a semi-structured think-aloud individual interview. Think-aloud interviews are likely to unveil underlying mental processes and knowledge structures, and therefore, are appropriate for a study that seeks to understand teachers' mental representations of the relationship between modeling and problem solving (Kelly & Lesh, 2000). Open-ended and broad prompts were used to elicit a wide range of productive thinking as naturally as possible. Each interview lasted for about an hour.

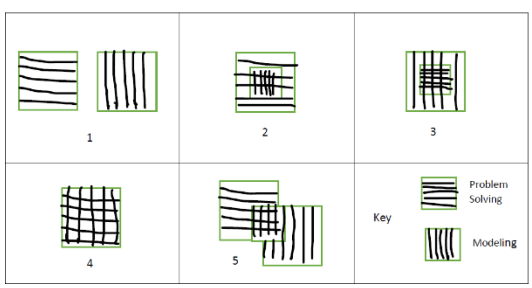

Figure 1: Graphic Representations of Various Relationships Between Modeling and Problem Solving

At the beginning of the interview, each teacher was given (a) Figure 1 and (b) transcript of their verbal response to the question: "What does mathematical modeling mean to you?" (At the end of the earlier study mentioned above). Each teacher was asked to choose one representation

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or draw an alternative one that best captured their understanding of the relationship based on their response to the question above, and then explain and justify their choice. Each teacher was also given the chance to change their choice based on their thinking at the time of the interview, an opportunity that none of the eight teachers took. In addition, each teacher was asked to explain how their choice might influence their perceptions of teaching mathematical modeling in 6-12 mathematics classrooms.

Data Analysis

A hybrid approach of inductive (emergent) and deductive (theoretically guided) coding (Fereday & Muir-Cochrane, 2006) was used to analyze the data. Alexander and Dochy's (1995) graphic representations and the corresponding categories existed *a priori* and were applied to the data directly as provisional codes. However, codes that emerged during the analysis were also allowed to avoid premature closure, and to uncover new ideas (Saldaña, 2016; Strauss & Corbin, 1990). Several rounds of coding were performed on the data. The first round of coding focused on identifying and labeling excerpts of interest. The second round of coding refined the initial codes as guided by the research questions. Codes that were considered as trivial or less relevant to the study were removed. As analysis continued, codes that shared similar attributes were grouped together into categories or themes during the third round of coding. Finally, a fourth round of coding was conducted to make appropriate modifications to achieve a better fit between the final codes and the data and to make sure that the coding categories had stabilized.

Results

Teachers' Conceptualizations of the Relationship between Modeling and Problem Solving

Each of the final coding categories and sample responses from the eight teachers are presented in Table 1. Alexis was the only teacher who chose Option 1. She was hesitant to claim that the two processes are completely unrelated, but she was certain that they are distinct from each other due to the very different natures of the two processes. Although problem solving has an obvious path; with modeling, the learner needs to find the "entire mathematics approach."

Brian and Alicia both chose option 2. For both teachers, problem solving is the bigger purpose. Modeling also serves the purpose of solving problems. When asked how traditional application problems are different than MEAs, Brian explained that traditional application problems are "very basic," then he shared the example of giving students a constant speed of a train and a travel distance, and ask them: "How long is it going to take to go this far?" On the other hand, Brian viewed MEAs as more complicated: "take a step outside of this [real-life scenario], and we can start applying some math concepts to it, then step back into our real-world scenario" to find out whether a solution "fits the bill." Alicia described MEAs as requiring "more depth of knowledge," and "really, really difficult, like having them [students] think outside of the box." For both teachers, the larger box contains all kinds of problem solving, including context-free problems, such as solving or manipulating an equation. Applied problem solving that involves real-world contexts belongs to the box in the middle. MEAs or other advanced modeling activities also belong to the box in the middle and constitute a more difficult subset [not shown on the figure] of all applied problem solving.

Ann was the only teacher who chose Option 5. She explained that the part that belongs to only problem solving but not modeling represents solving "cut and dry equations" using a given formula [i.e., context-free problems]. The part that belongs to modeling but not problem solving are problems or activities like MEAs that are "open-ended" and "extensive," and may not always

lead to an accurate answer. The part where modeling and problem-solving intersect must "making sense of real-world situations" and require interpretation before coming up with a solution, but still must have an "accurate" answer or "one true answer."

Sarah chose Option 3. Carrie was debating between Option 3 and Option 4. Both teachers believed that problem solving in a mathematics classroom must involve mathematical symbols or expressions; and therefore, only after a real-world situation is mathematized, problem solving begins. The steps before mathematization such as interpreting and visually representing a scenario are part of modeling or preparation for problem solving but not problem solving.

Katherine and Eric chose Option 4 as their best representation of the relationship between modeling and problem solving. These teachers emphasized the cyclic and iterative nature of the modeling process and how problem solving, and modeling are integrated into the same process and are mutually dependent on each other. Carrie (who chose both Option 3 and Option 4) also pointed out even "simple little problem" like 3 + 2 can be a mathematical representation of a real-world scenario and therefore, math is all about describing the real world. Eric shared a similar view and argued that addition and subtraction in early grades, as well as using multivariate functions to represent and predict an economic phenomenon in econometrics are all mathematical modeling except for their "varying degrees of difficulty and fluidity."

Table 1: Sample Responses for Each of the Coding Categories

Coding Categories	Sample Responses
Straightforward versus open-ended	Classroom-based problems, I'm literally only looking for a solution. I am not really looking for the in between…simply do what the formula is asking me to do…whereas with modeling…still given a given, but you have to produce the goal, and you have to produce everything in between, like your approach to how to get your solution. (Alexis, Option 1)
Modeling as a tool for problem solving	Modeling can be used for problem solving. (Brian, Option 2) The whole purpose of modeling is to assist with problem solving… Problem solving would be the bigger picture. (Alicia, Option 2)
Overlapping but different	The problem solving [part], I feel like is just like handing them an equation to solve, that's problem, and it's got to be right or wrong answer… the modeling part I kind of…realized that's even more real-world type situations in context … A lot of times, one student may interpret it differently than another student. They're not necessarily always getting the same answer…Modeling is not always accurate, I guess, because you can get, you know, different opinions from groups or students…modeling could help solve the problem, so maybe that would be the middle…[but] also has to be accurate. (Ann, Option 5)
Problem solving embedded in modeling	Problem solving is embedded within modeling. (Sarah, Option 3) There's more to modeling the real world than just problem solving. (Carrie, Option 3 & 4)

Modeling and problem solving as an integrated process	Can’t have one without the other…Modeling is just like math in general…used to describe the real world…you can’t have modeling without math, and you can’t really have math without modeling. (Carrie, Option 3 & 4) Enmeshed with each other. (Katherine, Option 4) They're one in the same…varying degrees of difficulty and fluidity. (Eric, Option 4)

Teaching Mathematical Modeling in Mathematics Classrooms

In response to the question: “How do you think your conceptualization of the relationship between modeling and problem solving influences your decisions regarding teaching mathematical modeling in mathematics classrooms,” the eight teachers focused on two dimensions of instruction: instructional sequence and problem/activity types (see Figure 2). Among all the teachers, Alexis was the most hesitant to teach mathematical modeling to her students. She felt that she was not prepared to enact modeling activities in her classrooms and emphasized that formal training was needed before she would be ready. She was also concerned about her students, stressing: “I don’t feel like everybody will be able to grasp the concept [of mathematical modeling].” She admitted that she was more comfortable with the traditional instructional sequence in which formal demonstration of mathematical concepts should happen before asking students to use the concepts to solve traditional application problems.

Instructional Sequence / Problem /Activity Types	Traditional Instructional Sequence (Demonstration -> Application Problems)	Modeling Approach (Modeling with Problem Solving -> Direct Instruction)
Traditional Applied Problems	Alexis	Brian, Alicia
MEAs or Interdisciplinary Projects		Ann, Katherine, Eric, Sarah, Carrie

Figure 2: Teachers’ Preferred Instructional Sequences and Types of Problems

The rest of the teachers all shared the belief that teaching mathematical modeling was both important and lacking in secondary mathematics education. In addition, they all preferred a modeling approach to teaching mathematics, which is, having students engage and struggle with a mathematical modeling activity first (to develop mathematical concepts naturally) before formally introducing mathematical concepts. Especially Alicia, when reflecting on the changes in her own instructional sequence, stated “introducing a model first could be really beneficial for the students.” With the traditional approach, Alicia observed, “they [her students] were lost,” and “they’re going through the motions, and they don’t really understand the concept of the function.” However, both Brian and Alicia were also satisfied with using traditional application problems as modeling activities.

The rest of the teachers were more open to activities like MEAs. Katherine acknowledged that although she was using a pilot program that required more problem solving from her students than before, the problems were still just requiring students to consider one variable at a time; MEAs require the learner to consider multiple factors at the same time and ideally should be used instead. Ann and Eric both felt positively about how the math teams at their schools were regrouping their standards so that one project can address multiple standards at the same time.

All three teachers pointed out (a) the benefits of working with colleagues in other disciplines, and (b) the similarities between MEAs and the interdisciplinary projects at their own schools.

Discussion and Conclusion

The eight teachers' conceptualizations of the relationship between modeling and problem solving are diverse and mostly consistent with research literature. For example, Alexis's focus is on whether the path from givens to goals is obvious or hidden. This is also pointed out by Lesh and colleagues as one of the key differences between traditional application problems and MEAs (Lesh et al., 2000). Brian's and Alicia's view of modeling as a subset of problem solving is consistent with what Lesh and colleagues (Lesh et al., 2000; Lesh & Doerr, 2003) call "the traditional view." This view considers all application problems, including MEAs, as a subset of all kinds of problem solving. Ann's review is more consistent with the MMP advocated by Lesh

and colleagues (Lesh et al., 2000; Lesh & Doerr, 2003). If we do not consider the part in Option 5 of Figure 1 that belongs to only problem solving but not modeling (i.e., according to Ann, context-free problems that require using formulas only), the rest of the graphic representation is consistent with MMP in terms of viewing traditional application problems as a subset of modeling activities in general. Sarah's and Carrie's view of solving traditional application problems as a reduced process of the modeling cycle is also consistent with Blum and Niss (1991). The view that modeling and problem solving are completely enmeshed, inseparable and mutually dependent on each other is a new conceptualization not found in research literature.

There is some preliminary evidence that a teacher's conceptualization of the relationship between modeling and problem solving is likely to be related to the instructional sequence and type of problems they prefer. For example, one teacher's (Alexis) view seems to create a barrier between the two processes, which makes the teacher feel the transition from the traditional approach to the modeling approach more challenging. In addition, when modeling is viewed as a more advanced form of problem solving, MEAs are often seen as another subset of all applied problems. This view does not seem to deter a teacher (e.g., Alicia) from successfully adopting the modeling approach or teaching mathematical modeling. However, under this view only traditional application problems (e.g., use an initial fee and fixed monthly payment to model linear functions) are likely to be used to teach mathematical modeling. Teachers who view modeling and problem solving as two integrated and mutually dependent processes or whose view is consistent with the MMP are more open to not only the modeling approach but also teaching mathematical modeling using open-ended, complex activities.

In conclusion, this study is an exploratory study that involves only eight teachers. Although the eight teachers' conceptualizations of the relationships between modeling and problem-solving are diverse, it is possible that other conceptualizations are missed. The relationship between these conceptualizations and instructional preferences is tentative due to the small number of teachers studied and needs further research. Despite these limitations, this study offers some implications for how to increase the use of modeling to teach mathematics in 6-12 classrooms. First, the view that mathematical modeling and problem solving are two distinct processes seems to be the least ideal for encouraging modeling in the mathematics classroom. In this study, the teacher who held this view was the most hesitant to teach modeling because she perceived considerable risks for using modeling in her teaching. Second, all the other conceptualizations including (a) modeling as a subset of problem solving, (b) the two are overlapping, (c) problem solving is embedded in modeling, and (d) modeling and problem solving is an integrated process, seem to be compatible with the instructional approach and sequence that integrate modeling throughout the curriculum rather than saving modeling for the end of a unit or a course. This instructional approach and sequence often allow students to

explore mathematics problems collaboratively before direct instruction and has shown to be more effective than the traditional approach with teacher demonstration followed by practicing traditional problems (Boaler, 2015). Third, some teachers may hold "the traditional view" that all application problems, including MEAs, constitute a subset of all kinds of problem solving, and therefore, do not make a clear distinction between traditional application problems and MEAs.

These teachers may feel content with using traditional application problems even if these problems often focus on developing procedural skills rather than reasoning ability. Teacher educators may steer teachers away from conceptualizations that are counterproductive to the use of modeling activities that require higher-order thinking and help teachers develop MMP by building upon productive conceptualizations.

References

Aguilar, J. J. (2021). Modeling through model-eliciting activities: An analysis of models, elements, and strategies in high school. The cases of students with different level of achievement. *Mathematics Teaching Research Journal, 13*(1), 52-70.

Alexander, P. A., & Dochy, F. J. R. (1995). Conceptions of knowledge and beliefs: A comparison across varying cultural and educational communities. *American Educational Research Journal, 32*(2), 413-442.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modeling problems? In C. P. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modeling (ICTMA 12): Education, engineering and economics* (pp. 222-231). Chichester, UK: Horwood.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modeling, applications, and links to other subjects: State trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*, 37-68.

Boaler, J. (2015). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching* (1st ed.). Jossey-Bass.

Burkhardt, H., & Pollak, H. (2006). Modeling in mathematics classrooms: Reflections on past developments and the future. *ZDM: The International Journal on Mathematics Education, 38*(2), 178-195. https://doi.org/10.1007/BF02655888

Bybee, R. W. (2010). Advancing STEM education: A 2020 vision. *Technology and Engineering Teacher, 30*, 30-35. Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Borba, M., Geiger, V., Stillman, G., English L. D., Wake, G.,

Kaiser, G., & Kwon, O. N. (2014). Mathematical modeling in school education: Mathematical, curricular, cognitive, instructional, and teacher education perspectives. In P. Liljedahl, C., Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education [PME]*, vol. 1 (pp. 145-172). Vancouver, CA: PME.

Doerr, H. M. (2007). What knowledge do teachers need for teaching mathematics through applications and modeling? In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modeling and applications in mathematics education: The 14th ICMI Study, New ICMI Studies Series no. 10* (pp. 69-78). Springer.

Doerr, H. M. (2016). Designing sequences of model development tasks. In C. R. Hirsch, & A. R. McDuffe (Eds.), *Mathematical modeling and modeling mathematics* (pp. 197-206). The National Council of Teachers of Mathematics [NCTM].

Doerr, H., M. & English, L. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education, 34*, 110-136. https://doi.org/10.2307/30034902

Dossey, J. (2010). Mathematical modeling on the catwalk: A review of modeling and applications in mathematics education: The 14th ICMI Study. *Journal for Research in Mathematics Education, 41*(1), 88-95.

Fereday, J., & Muir-Cochrane, E. (2006). Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *International Journal of Qualitative Methods, 5*(1), 80-92. https://doi.org/10.1177/160940690600500107

Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modeling process. *ZDM: The International Journal on Mathematics Education, 38*(2), 143-162. https://doi.org/10.1007/BF02655886

Kelly, A. E., & Lesh, R. A. (Eds.). (2000). *Handbook of research design in mathematics and science education*. Routledge.

Lesh, R. A., & Doerr, H. M. (Eds.) (2003). *Beyond constructivism: A models and modeling perspectives on mathematical problem solving, learning, and teaching*. Lawrence Erlbaum Associates.
Lesh, R. A., & Harel, G. (2003). Problem solving, modeling, and local conceptual development. *Mathematical Thinking and Learning, 5*(2-3), 157-189. https://doi.org/10.1080/10986065.2003.9679998
Lesh, R. A., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591–646). Lawrence Erlbaum Associates, Inc.
Lesh, R. A., & Lehrer, R. (2003). Models and modeling perspectives on the development of students and teachers. *Mathematical Thinking and Learning, 5*(2-3), 109-12. https://doi.org/10.1080/10986065.2003.9679996 Lesh, R. A., & Yoon, C. (2007). What is distinctive in (our views about) models and modeling perspectives on
mathematical problem solving, learning, and teaching? In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modeling and applications in mathematics education: The 14th ICMI Study, New ICMI Studies Series no. 10* (pp. 161-170). Springer.
Lesh, R. A., & Zawojewski, J. S. (2007). Problem solving versus modeling. In R. A. Lesh, P. L., Galbraith, C. R. Haines, & A. Hurford (Eds.), Modeling students' mathematical modeling competencies (pp. 237-243). Springer.
Pollak, H. O. A. (2003). A history of the teaching of modeling. In G. M. A. Stanic, & J. Kilpatrick, (Eds.), A history of school mathematics (pp. 647-669). National Council of Teachers of Mathematics [NCTM].
Saldaña, J. (2016). The coding manual for qualitative researchers (3rd ed.). Sage.
Strauss, A., & Corbin, J. (1990). Basics of qualitative research: Grounded theory procedures and techniques (2nd ed.). Sage.
Selden, A., Selden, J., Hauk, S., & Mason, A. (1999). Do calculus students eventually learn to solve non-routine problems? Technical report. No. 1999-5. Cookeville, TN: Tennessee Technological University.
Zawojewski, J. S. (2013). Problem solving versus modeling. In R. A. Lesh, P. L., Galbraith, C. R. Haines, & A. Hruford (Eds.), Modeling students' mathematical modeling competencies (pp. 237-243). Springer.
Zbiek, R. M. (2016). Supporting teachers' development as modelers and teachers of modelers. In C. R. Hirsch, & A.R. McDuffe (Eds.), Mathematical modeling and modeling mathematics (pp. 263-272). The National Council of Teachers of Mathematics [NCTM].
Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understanding of curricular mathematics. Educational Studies in Mathematics, 63(1), 89-112. https://doi.org/10.1007/s10649-005-9002-4